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ON AUSLANDER-REITEN COMPONENTS AND SIMPLE MODULES FOR FINITE GROUPS OF LIE TYPE

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Introduction

Let kG be the group algebra of a finite group G over an algebraically closed field k of characteristic p, where p is a prime.

Let B be a p-block of kG with defect group $\delta(B) =_G D$. Then B is called to be of wild representation type, if D is neither cyclic, nor dihedral, semi-dihedral or a generalized quaternion group.

Let Θ be a connected component of the stable Auslander-Reiten quiver $\Gamma_s(B)$ of a wild p-block B of kG. In [6] K. Erdmann has shown that the tree class of Θ is A_∞. It is therefore natural to ask where a simple kG-module S of a wild p-block B can be found in its Auslander-Reiten component $\Theta(S)$.

The first author proved in [11] that a simple kG-module S of a wild p-block B must lie at the end of $\Theta(S)$ if G is p-solvable. Moreover, he showed that, if there is some simple module which does not lie at the end, then there exist several simple modules lying at the end and having uniserial projective covers. We hope that this result may give a device for determining where simple modules lie in $\Gamma_s(B)$.

In this paper, we prove a similar assertion for wild p-blocks of a finite group G of Lie type, when p is the defining characteristic of G. Our main result is as follows.

Theorem. *Let G be a finite group of Lie type defined over a field K of characteristic p. Let B be a block of G with full defect of wild representation type. Then any simple kG-module S of B lies at the end of its Auslander-Reiten component $\Theta(S)$.*

It is not always true that simple modules S of wild blocks B lie at the end of $\Theta(S)$. Using the decomposition matrix of [10], we show that a 5-block of $F_4(2)$ with full defect has a simple module with dimension 875823, which lies in the second row from the end.

This paper is organized as follows. We review several result in Auslander-Reiten theory in Section 2. In Section 3, we recall results on the vertices, sources of simple kG-modules for a finite group G of Lie type. The proof of the main theorem is found in Section 4. Concerning our terminology and notation we refer to the books by Benson [1], Erdmann [4] and Feit [7].

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1. Preliminary results of the Auslander-Reiten theory

In this section, we recall several results of the Auslander-Reiten theory and prove a proposition, which is needed later on.

For any non-projective indecomposable kG-module M, there exists an Auslander-Reiten sequence

$$\mathcal{A}(M) : 0 \to \Omega^2 M \to X_M \to M \to 0,$$

unique up to isomorphism of short exact sequences, where the middle term X_M is a well-determined kG-module, and Ω denotes the Heller operator.

In the Auslander-Reiten theory Ω plays an important role. We will freely use the following standard facts:

(1.1) For a subgroup H of G, a kG-module M, and a kH-module L, we have

$$\Omega^i(M)_H \cong \Omega^i(M_H) \oplus X_i, \quad \Omega^i(L)^G \cong \Omega^i(L^G) \oplus Y_i$$

for all integers i, where X_i and Y_i are zero or projective kH-module and kG-module, respectively.

(1.2) For a non-projective simple kG-module S with projective cover P_S

$$\mathcal{A}(\Omega^{-1}(S)) : 0 \to \Omega(S) \to P_S \oplus P_S J/S \to \Omega^{-1}(S) \to 0$$

is the Auslander-Reiten sequence of $\Omega^{-1}(S) = P_S/S$, where J denotes the Jacobson radical of the group algebra kG.

The stable Auslander-Reiten quiver $\Gamma_s(kG)$ of kG is a directed graph, whose points are parameterized by isomorphism classes of non-projective indecomposable kG-modules. If a non-projective indecomposable kG-module N appears as a direct summand of X_M with multiplicity r, then we write r arrows from N to M and $\Omega^2 M$ to N. Let Γ be a connected component of $\Gamma_s(kG)$. Then it is known that Γ has a subquiver T such that all the modules in Γ can be written as $\Omega^{2i} M$ for some integer i and M in T. This T is determined uniquely up to quiver isomorphisms and called the tree class of Γ. For group algebras, T is either a finite or infinite Dynkin or Euclidean diagram by Theorem 2.31.2 of Benson [1], p. 160. However, for blocks of wild representation type, Theorem 1 of K. Erdmann [6] asserts that each connected component of $\Gamma_s(B)$ has tree class A_∞.

Concerning the restriction of an Auslander-Reiten sequence, we prove the following.

Proposition 1.3. *Let M be a non-projective indecomposable kG-module with vertex $vx(M) =_G P$, a Sylow p-subgroup of G. If the restriction M_P is indecomposable, then the restriction $\mathcal{A}(M)_P$ of the Auslander-Reiten sequence $\mathcal{A}(M)$ is isomorphic to a direct sum of $\mathcal{A}(M_P)$ and a split short exact sequence $0 \to X \to X \to 0 \to 0$, where X is zero or a projective kP-module. Moreover, if X_M has l non-projective indecomposable direct summands, then X_{M_P} has at least l non-projective direct summands.*

Proof. Let $H = N_G(P)$. Note that M_H is the Green correspondence of M with respect to (G, P, H). It follows from an exercise of [1], p. 93 that $\mathcal{A}(M)_H$ is isomorphic to a direct sum of $\mathcal{A}(M_H)$ and some short exact sequence. Moreover, by Proposition 7.9 of [8], $\mathcal{A}(M_H)_P$ is isomorphic to a direct sum of Auslander-Reiten sequences and possibly a split exact sequence. Thus $\mathcal{A}(M_P)$ appears as a direct summand of $\mathcal{A}(M)_P$. However, since M_P is indecomposable and since the first term of $\mathcal{A}(M)_P$ is isomorphic to $\Omega^2(M_P) \oplus X$, where X is zero or projective, the first assertion follows.

Suppose that $X_M = N_1 \oplus \cdots \oplus N_l \oplus N'$, where N_i are non-projective indecomposable kG-modules. Then $(N_i)_P$ has at least one non-projective indecomposable direct summand by Green's vertex theory.

Since $(X_M)_P \cong X_{M_P} \oplus X \cong \bigoplus_{i=1}^{l} (N_i)_P \oplus N'_P$, the kP-module X_{M_P} has at least l non-projective indecomposable summands. □

2. Sources of simple modules of finite groups of Lie type

In this section, we recall several facts on simple modules of finite groups G of Lie type over an algebraically closed field k of characteristic p, where p is the defining characteristic of G.

The following result is due to Dipper [2], [3]. It is restated for the sake of completeness.

Proposition 2.1. *Let G be a finite group of Lie type and k an algebraically closed field of characteristic p, where p is the defining characteristic of G. Let S be a non-projective simple kG-module. Then the following assertions hold*:

(i) *A vertex of S is a Sylow p-subgroup P of G.*

(ii) *The restriction S_P is a source of S.*

(iii) *The module S_P has a 1-dimensional socle and a 1-dimensional head.*

Proof. It suffices to show the assertion for perfect finite groups of Lie type. The first and the second statement can be found in Theorem 4.5 of [3] and the main theorem of [2].

The third statement is proved in Corollary 1.9 of [2] for untwisted groups of Lie type. The twisted cases can be treated similarly by using the relevant subsidiary results appearing in the sections 3 to 6 of [3]. It follows then by the argument of Lemma 1.7 of [2] that S_P is a cyclic module. Thus S_P has a 1-dimensional head. Since the dual kG-module S^* is also non-projective and simple, S_P^* has also a 1-dimensional head. Hence S_P has a 1-dimensional socle.

3. Proof of Theorem

Let S be a simple kG-module which is not at the end of its Auslander-Reiten component. Then the middle term X of its Auslander-Reiten sequence $\mathcal{A}(S)$ contains at least two non-projective indecomposable direct summands N_i. Thus $X \cong \bigoplus_{i=1}^{l} N_i \oplus Q$, where $l \geq 2$, and Q is a projective or the zero kG-module.

By Dipper's theorem stated as Proposition 2.1 the restriction S_P of S to a Sylow p-subgroup P of G has a simple socle and a simple head. In particular, S_P is indecomposable. Now Proposition 1.3 asserts that the middle term X_{S_P} of the Auslander-Reiten sequence $\mathcal{A}(S_P)$ has $l \geq 2$ non-projective indecomposable direct summands. Hence S_P is not at the end of its Auslander-Reiten component.

However as B and therefore kP is of wild representation type, the Auslander-Reiten component Θ of S_P is of tree class A_∞ by Theorem 1 of Erdmann [6]. Therefore Lemma 1.4 of Erdmann [5], p. 374 asserts that S_P lies at the end of Θ, because the kP-module S_P has simple head and simple socle. This is a contradiction.

4. Examples of simple modules not lying at the end

According to the 5-modular decomposition matrix for $F_4(2)$ given by Hiss [10], Theorem 2.5, the group $F_4(2)$ has a simple module S with dimension 1326 such that its projective cover P_S has the Loewy series

$$\begin{matrix} S \\ T \\ S \end{matrix}$$

where T is a simple module with dimension 875823. Hence

$$\mathcal{A}(\Omega^{-1}(S)) : 0 \to \Omega(S) \to P_S \oplus T \to \Omega^{-1}(S) \to 0$$

is the Auslander-Reiten sequence of $\Omega^{-1}(S)$. Therefore T lies in the second row from the end of its Auslander-Reiten component $\Theta(T)$.

REMARK. A similar example is found for the covering group $2.Ru$ of the sporadic

Rudvalis simple group. By Hiss [9], there is a 3-modular simple module with dimension 10528 which also lies in the second row of its Auslander-Reiten component.

REMARK. At the moment, we do not know a wild block which has a simple module lying in the n-th row from the end for some n with $n \geq 3$.

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